Noncommutative Geometry of Discrete Groups

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In mathematics, groups naturally arise as symmetries of geometric objects. The structure of a group is reflected in its representation theory. The set of all representations of a group forms the dual space of the group. In the case of an abelian discrete group, the dual space is a compact Hausdorff space that completely characterizes the original group and can be understood by tools from algebraic topology and classical analysis. However when the group is nonabelian, its dual space often fails to be Hausdorff and is generally ill-behaved. Noncommutative geometry is a branch of mathematics that, in particular, provides novel tools to study the space of representations for general discrete groups [12]. This is achieved by studying certain noncommutative geometric objects associated to discrete groups.

Recall that a (complex) representation of a group Γ is given by a group homomorphism $\pi : \Gamma \to GL(V)$, where $GL(V)$ is the group of all invertible linear maps on a complex vector space V. Given a finite group Γ, the collection of isomorphism classes of finite dimensional representations naturally generates the so-called representation ring of Γ, denoted by $R(\Gamma)$. For example when $\Gamma = \{e\}$ is the trivial group, then $R(\{e\})$ is just $\mathbb{Z}$. This is simply a restatement of the fact that a representation of the trivial group is completely determined by its dimension. In this sense, the ring $R(\Gamma)$ is just a generalization of the notion of dimension for finite groups.

In general, however, when Γ is infinite, the representation ring $R(\Gamma)$ as defined above is too algebraic to capture enough information about the representations of Γ. Instead, we need to introduce some analysis. More precisely, let $\ell^2(\Gamma)$ be the Hilbert space of ℓ^2-summable functions on Γ. The group Γ acts on $\ell^2(\Gamma)$ by translations, so that $\ell^2(\Gamma)$ becomes a representation of Γ, called the regular representation. Moreover the action of Γ extends linearly to an action of the group algebra $\mathbb{C}[\Gamma]$, where $\mathbb{C}[\Gamma]$ consists of all formal sums $\sum_{\gamma\in\Gamma} a_\gamma\gamma$ such that $a_\gamma = 0$ for all but finitely many $\gamma \in \Gamma$, with multiplication

$$\Big(\sum_{\gamma\in\Gamma} a_\gamma\gamma\Big)\cdot\Big(\sum_{\alpha\in\Gamma} b_\alpha\alpha\Big) = \sum_{\beta\in\Gamma}\Big(\sum_{\gamma\alpha=\beta} a_\gamma b_\alpha\Big)\beta.$$

Let $B(\ell^2(\Gamma))$ be the algebra of all bounded linear operators on $\ell^2(\Gamma)$; we define the reduced group C^*-algebra of Γ, denoted by $C_r^*(\Gamma)$, to be the operator-norm closure of $\mathbb{C}[\Gamma]$ in $B(\ell^2(\Gamma))$.

Now, the reduced group C^*-algebra $C^*(\Gamma)$ is precisely the noncommutative geometric object associated to Γ that we referred to at the beginning. Intuitively speaking, if we think of an algebra as the algebra of functions on some topological space, then the maximal ideals of that algebra correspond to the points of the topological space. When Γ is abelian, $C_r^*(\Gamma)$ has many maximal ideals; indeed, in this case, the set of maximal ideals forms a nice Haudorff space, which is, in fact, just the dual space of Γ we mentioned earlier. However, for many nonabelian discrete groups such as the free group F_n with $n \geq 2$ generators, $C_r^*(\Gamma)$ turns out to be simple, that is, it does not admit any nontrivial ideals [36]. We see that the concept of points no longer makes sense for $C_r^*(\Gamma)$ in general. This is a completely new phenomenon that only exists in the noncommutative world!

Algebraic topology teaches us that cohomology theory provides some of the most fundamental invariants for understanding geometric objects. However, classical singular or cellular cohomology theory does not extend to the noncommutative geometric setting. The only cohomology theory that works for both commutative and noncommutative geometric objects is called K-theory; informally, it is a generalization of the notion of representation ring. Let us recall the definition of the K-theory group, cf. [9]. Let $A = C_r^*(\Gamma)$ and $M_\infty(A) = \bigcup_n M_n(A)$, where $M_n(A)$ is the algebra of all $n \times n$ matrices with coefficients in A. Let $V(A)$ be the set of all equivalence classes of idempotents in $M_\infty(A)$, where two idempotents p and q are said to be equivalent if there exists an invertible element $u \in M_\infty(A)$ such that $upu^{-1} = q$. Observe that $V(A)$ is an abelian semigroup with addition

$$[p_1] \oplus [p_2] = \begin{bmatrix} p_1 & 0 \\ 0 & p_2 \end{bmatrix}.$$

Recall that there is a natural abelian group, called the Grothendieck group, associated to each abelian semigroup. More precisely, if $(S, +)$ is an abelian semigroup, then the Grothendieck group of S is defined to be $S \times S/\sim$, where $(s_1, s_2) \sim (t_1, t_2)$ if $s_1 + t_2 + s = s_2 + t_1 + s$ for some $s \in S$. For example, if $S = \mathbb{N}$, then the corresponding Grothendieck group is $\mathbb{Z}$. The K-theory group $K_0(A)$ is defined to be the Grothendieck group of $V(A)$. In the case where Γ is finite, we have $C_r^*(\Gamma) = \mathbb{C}[\Gamma]$ and $K_0(\mathbb{C}[\Gamma]) \cong R(\Gamma)$.

In general, it is difficult to compute $K_0(C_r^*(\Gamma))$. In the case where Γ is abelian, we have an algorithm to compute $K_0(C_r^*(\Gamma))$, using tools from algebraic topology. This is because $C_r^*(\Gamma)$ admits many ideals in this case, which allows us to decompose $C_r^*(\Gamma)$ into tractable pieces. However, this method breaks down immediately in general, for, we have seen, $C_r^*(\Gamma)$ is often simple for nonabelian groups. An algorithm for computing $K_0(C_r^*(\Gamma))$ in the general case is envisioned by the Baum–Connes conjecture [4, 5]. In very rough terms, the Baum–Connes conjecture states that $K_0(C_r^*(\Gamma))$ is isomorphic to something that is completely topological, which can therefore be computed by tools from algebraic topology such as the Mayer–Vietoris sequence.

Let us give a more precise description of the Baum–Connes conjecture. For simplicity, let us assume that Γ is torsion-free from now on. Recall that the classifying space $B\Gamma$ of Γ is a topological space such that any Γ-covering space of a compact Hausdorff space X is uniquely determined (up to homotopy) by a continuous map $f : X \to B\Gamma$. For example, if $\Gamma = \mathbb{Z}^n$, then $B\Gamma = T^n$, the n-dimensional torus. To describe the Baum–Connes conjecture, we need to introduce the K-homology group $K_0(B\Gamma)$ of the classifying space $B\Gamma$ and the Baum–Connes assembly map $\mu : K_0(B\Gamma) \to K_0(C_r^*(\Gamma))$.

Let us first introduce the Dirac operators, cf. [2, 3, 29]. Roughly speaking, a Dirac operator on a manifold is a first order differential operator whose square is the Laplacian. For example, on $\mathbb{R}^1$, the standard Dirac operator is $D = \frac{1}{i}\frac{d}{dx}$ with its square $D^2 = -\frac{d^2}{dx^2} = \Delta$; on $\mathbb{R}^2$, the standard Dirac operator is

$$D = \begin{pmatrix} 0 & -\frac{\partial}{\partial x} + i\frac{\partial}{\partial y} \\ \frac{\partial}{\partial x} + i\frac{\partial}{\partial y} & 0 \end{pmatrix} \text{ with } D^2 = \begin{pmatrix} -\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} & 0 \\ 0 & -\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} \end{pmatrix}.$$

Observe that on $\mathbb{R}^2$, we need to use matrices in order to get a "square root" of the Laplacian. In general, there is an elegant way to use Clifford algebras to define the Dirac operator on $\mathbb{R}^n$. Now, we would like to know how to construct a Dirac operator on a manifold. Recall that a manifold is a collection of coordinate charts patched together. On each coordinate chart, which can be assumed to be $\mathbb{R}^n$, we have seen how to construct a Dirac operator. We would like to glue these locally defined Dirac operators together to obtain a Dirac operator on the whole manifold. It turns out that a manifold needs to satisfy a certain topological condition, called spin condition, for such a gluing to be possible (cf. [29, Appendix D]). Finally, the K-homology group $K_0(B\Gamma)$ is an abelian group whose elements consist of Dirac operators and their generalizations on closed manifolds, i.e., compact manifolds without boundary, subject to certain equivalence relations [6, 7, 8].

Having defined the K-homology group, let us now describe the Baum–Connes assembly map μ, which is sometimes also called the higher index map. First, we recall the classical Fredholm index theory. The Dirac operator D on a manifold M is an elliptic differential operator, which naturally induces a Fredholm operator on a Hilbert space H. By definition, a Fredholm operator F is a bounded linear operator on H with a finite dimensional kernel and a finite dimensional cokernel, where the cokernel of F is the quotient of H by the range of F. The index of F is defined to be the integer

$$\operatorname{Ind}(F) = \dim(\ker F) - \dim(\operatorname{coker} F).$$

By convention, we define the Fredholm index of D to be the index of the associated Fredholm operator F. The Fredholm index is a homotopy invariant. The celebrated Atiyah–Singer index theorem states that the index of the Dirac operator D on M can be expressed by cohomological geometric data of M [2, 3].

Observe that if we view the finite dimensional vector spaces $\ker F$ and $\operatorname{coker} F$ as representations of the trivial group $\{e\}$, then the Fredholm index is reminiscent of the notion of dimension of representations. This might sound a little ad hoc at the moment, but we shall justify it as follows. Consider the universal cover $\widetilde{M}$ of a closed manifold M and let Γ be the fundamental group of M. The Dirac operator D on M naturally lifts to a Dirac operator on $\widetilde{D}$ on $\widetilde{M}$. If Γ is finite, $\widetilde{M}$ is again closed. As we have seen, $\widetilde{D}$ naturally induces a Fredholm operator in this case. Moreover, we notice that $\widetilde{D}$ is Γ-equivariant, that is, $\widetilde{D}$ commutes with the action of Γ on $\widetilde{M}$. As a result, the kernel and cokernel of $\widetilde{D}$ are, in fact, finite dimensional representations of Γ, and their formal difference gives rise to an element in $R(\Gamma) = K_0(\mathbb{C}[\Gamma])$; we call this element the higher index of D. In general, when Γ is not finite, $\widetilde{M}$ is not closed. In this case, $\widetilde{D}$ does not define a Fredholm operator on a Hilbert space, and hence the usual Fredholm index does not make sense. Rather, we need to generalize the notion of Fredholm operator by taking Γ into account, leading to the notion of generalized Fredholm operators. Regardless of Γ being finite or not, $\widetilde{D}$ always naturally gives rise to a generalized Fredholm operator, whose higher index is an element of $K_0(C_r^*(\Gamma))$, denoted by $\mathrm{Ind}_\Gamma(D)$ [32, 23]. The higher index contains the Fredholm index as its 0-dimensional information [1], and hence is a generalization of the Fredholm index. In particular, the higher index of D is a homotopy invariant, and, moreover, an obstruction to the invertibility of $\widetilde{D}$. Finally, the Baum–Connes assembly map μ is the group homomorphism defined by mapping elements of $K_0(B\Gamma)$ to their corresponding higher index in $K_0(C_r^*(\Gamma))$. The Baum–Connes conjecture states that the assembly map μ is an isomorphism.

As one can see, the noncommutative geometry of discrete groups is tied ultimately with the study of the higher index of Dirac type operators. The latter is often called higher index theory, and it has many applications to geometry and topology. For example, it is a fundamental tool in the study of the following conjecture.

Conjecture (Gromov-Lawson Conjecture)**.** *A closed aspherical manifold does not admit a Riemannian metric of positive scalar curvature.*

An aspherical manifold is a manifold whose universal cover is contractible. For example, the n-dimensional torus T^n is aspherical for each $n \geq 1$. In fact, the following question was one of the main motivating examples behind the above conjecture.

Question. *For $n \geq 2$, does the n-dimensional torus T^n admit a Riemannian metric of positive scalar curvature?*

The answer is "no" for all $n \geq 2$. The case where $n = 2$ follows immediately from the Gauss–Bonnet theorem. The case where $n \geq 3$, which turned out to be surprisingly involved, was solved by Schoen–Yau [37, 38] and Gromov–Lawson [15] using different methods. We sketch a proof using higher index theory. First, we recall that the Dirac operator D associated to a given Riemannian metric on T^n satisfies the Weitzenböck–Lichnerowicz identity $D^2 = \Delta + \kappa/4$, where κ is the scalar curvature of the given metric [29, 30]. Now, this Weitzenböck–Lichnerowicz identity lifts to $\mathbb{R}^n = \widetilde{T}^n$, the universal cover of T^n; in other words, we have

$$\widetilde{D}^2 = \widetilde{\Delta} + \frac{\widetilde{\kappa}}{4},$$

where $\widetilde{D}$ (resp. $\widetilde{\Delta}$ and $\widetilde{\kappa}$) is the lift of D (resp. Δ and κ) from T^n to $\mathbb{R}^n = \widetilde{T}^n$. Now if $\kappa > 0$ everywhere on T^n, then $\widetilde{D}$ would be invertible, implying that the higher index $\mathrm{Ind}_\Gamma(D) = 0$, where $\Gamma = \mathbb{Z}^n$. However, by homotopy invariance, $\mathrm{Ind}_\Gamma(D)$ is independent of the choice of metric on T^n. Moreover, an explicit calculation shows that[1] $\mathrm{Ind}_\Gamma(D) \neq 0$ for D on T^n. Hence, by contradiction, we see that T^n does not admit a Riemannian metric of positive scalar curvature. Observe that it was essential to use the higher index instead of the Fredholm index here, since the Fredholm index $\mathrm{Ind}(D)$ on T^n is, in fact, zero.

In the above example, an essential ingredient of the proof is the nonvanishing of the higher index of the Dirac operator. This turns out to be a common theme in many applications of noncommutative geometry to classical topology and geometry. From this perspective, the strong Novikov conjecture provides an algorithm to verify whether or not a higher index vanishes.

In the last two decades, there have been major breakthroughs in the study of the noncommutative geometry of discrete groups, especially those related to the Baum–Connes conjecture [17, 28, 22, 10, 31, 33, 40, 27] and the Novikov conjecture [23, 14, 13, 39, 25, 26, 45, 46, 24, 11]. In particular, the Novikov conjecture has been proven to hold for groups that are coarsely embeddable into Hilbert space [46]. This includes all discrete subgroups of linear groups [16]. In recent years, there has also been considerable progress in the study of secondary higher index theoretical invariants, such as the higher rho invariant [19, 18, 20, 21, 35, 34, 43, 42, 44, 41]. Secondary invariants often appear when primary invariants, such as the higher index, vanish. For example, as we have seen above, if a spin manifold M is equipped with a Riemannian metric g of positive scalar curvature, then the higher index $\mathrm{Ind}_\Gamma(D)$ of its Dirac operator D is zero. However, there is a natural way to associate a secondary higher index-theoretical invariant to the pair (D, g). This invariant is called the higher rho invariant of the pair (D, g), denoted by $\rho(D, g)$. Unlike the higher index, the higher rho invariant can very well depend on the choice of metric. For example, for two positive scalar curvature metrics g_1 and g_2 on M, if $\rho(D, g_1) \neq \rho(D, g_2)$, then g_1 and g_2 can not be connected by a path of positive scalar curvature metrics on M. This gives some nice applications of

[1] We have restricted our discussion of the higher index to the case of $K_0(C_r^*(\Gamma))$, which corresponds to even dimensional manifolds. There is a complete analogue in the case of odd dimensional manifolds, where the higher indices lie in $K_1(C_r^*(\Gamma))$. All discussions above apply equally to the odd dimensional case as well.

the higher rho invariant to classical geometry and topology. For example, it can be applied to study the topology of the moduli space of positive scalar curvature metrics on a given spin manifold [34, 35, 42]. Moreover, similar secondary invariants also lead to very interesting results in the study of the rigidity problem for topological manifolds [41].

The study of secondary invariants is ultimately related to the Baum–Connes conjecture. Secondary invariants can potentially be used to construct exotic elements in $K_0(C_r^*(\Gamma))$ that are not in the image of the Baum–Connes assembly map. For instance, given two positive scalar curvature metrics on an odd dimensional spin manifold, one can construct a secondary invariant in $K_0(C_r^*(\Gamma))$. It is an open question whether or not this secondary invariant is in the image of the Baum–Connes assembly map [44]. A better understanding of secondary invariants certainly will reveal new phenomena in the noncommutative geometry of discrete groups.

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